Kinetic Integral Solutions of the Boltzmann Equation*

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A kinetic picture is presented from which follow the time-dependent Boltzmann equation in the relaxation-time approximation and an exact solution in the form of a kinetic integral. As a special case of the latter, one obtains Chambers' integral solution of the time-independent equation. Confusion in the literature regarding solutions of the exact versus linearized equations is clarified.

WE start with the following equation

$$f(\boldsymbol{\varrho},\, t+dt)=f(\boldsymbol{\varrho}-\dot{\boldsymbol{\varrho}}dt,\, t)+\frac{dt}{\tau(\boldsymbol{\varrho})}[f^0(\boldsymbol{\varrho})-f(\boldsymbol{\varrho},t)], \quad (1)$$

where $f(\boldsymbol{\varrho},t)$ is the probability at time t of finding a particle at point $\boldsymbol{\varrho}=\mathbf{k},\,\mathbf{r}$ in phase space, i.e., with wave number $\mathbf{k}$ and position $\mathbf{r}$. Along with the dynamical law of motion,

$$\dot{\boldsymbol{\varrho}}=\dot{\mathbf{k}},\dot{\mathbf{r}}=(q/\hbar)[\mathbf{E}(\mathbf{r},t)+\mathbf{v}(\mathbf{k})\times\mathbf{B}(\mathbf{r},t)],\ \mathbf{v}(\mathbf{k})\,, \quad (2)$$

the first two terms of (1) reflect particle motion along a deterministic trajectory. The last term follows from a kinetic picture in which, in every time interval τ, a collision occurs which removes a particle from the probability distribution f and scatters it into the special equilibrium distribution f^0 (presumed known). Equation (1) immediately yields the time-dependent Boltzmann partial differential equation in the relaxation approximation:

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \boldsymbol{\varrho}}\cdot\dot{\boldsymbol{\varrho}}=-\frac{(f-f^0)}{\tau}. \quad (3)$$

However, if we fix attention on a point in phase space which moves along a *characteristic trajectory* defined by

$$d\boldsymbol{\varrho}=\dot{\boldsymbol{\varrho}}dt\,, \quad (4)$$

then the f corresponding to this moving point changes only via collisions; in fact, we obtain the ordinary differential equation

$$df/dt=-(f-f^0)/\tau\,, \quad (5)$$

which is readily integrated from time t' to a later time t to yield

$$f(\boldsymbol{\varrho},t)=f(\boldsymbol{\varrho}',t')\exp\left(-\int_{t'}^{t}\frac{dt'''}{\tau'''}\right)+\int_{t'}^{t}\frac{f^0(\boldsymbol{\varrho}'',t'')}{\tau''}\exp\left(-\int_{t''}^{t}\frac{dt'''}{\tau'''}\right)dt''. \quad (6)$$

In this equation, the points $\boldsymbol{\varrho}''$, t'', and the end point $\boldsymbol{\varrho}$, t, form the trajectory determined by the initial point $\boldsymbol{\varrho}'$, t', and by (2) and (4); τ'' is a shorthand notation for $\tau(\boldsymbol{\varrho}'')$. Equation (6) is a solution of (3) in the sense that, given the initial probability distribution $f(\boldsymbol{\varrho},t')$, the determination of $f(\boldsymbol{\varrho},t)$ is reduced to quadrature.

If we let t' recede into the infinite past, $f(\boldsymbol{\varrho},t)$ loses its dependence on the initial condition and approaches the steady-state distribution

$$\Phi(\boldsymbol{\varrho})=\int_{-\infty}^{t}\frac{f^0(\boldsymbol{\varrho}'',t'')}{\tau''}\exp\left(-\int_{t''}^{t}\frac{dt'''}{\tau'''}\right)dt''. \quad (7)$$

Equation (7) was originally suggested by Chambers[1] and has been verified (for a special case) by Budd[2] and, in general, by Tavernier,[3] who show directly that Φ is indeed an exact solution of the time-*independent* Boltzmann equation.

To clarify a certain confusion in the literature regarding solutions of the *linearized* Boltzmann equation, we now integrate (6) and (7) by parts. After rearranging, and defining δf and $\delta\Phi$ as deviations from equilibrium, i.e., $\delta f\equiv f-f^0$ and $\delta\Phi\equiv\Phi-f^0$, we obtain

$$\delta f(\boldsymbol{\varrho},t)=\delta f(\boldsymbol{\varrho}',t')\exp\left(-\int_{t'}^{t}\frac{dt'''}{\tau'''}\right)-\int_{t'}^{t}\frac{df^0}{dt''}(\boldsymbol{\varrho}'',t'')\exp\left(-\int_{t''}^{t}\frac{dt'''}{\tau'''}\right)dt'' \quad (6a)$$

and

$$\delta\Phi(\boldsymbol{\varrho})=-\int_{-\infty}^{t}\frac{df^0}{dt''}(\boldsymbol{\varrho}'',t'')\exp\left(-\int_{t''}^{t}\frac{dt'''}{\tau'''}\right)dt''. \quad (7a)$$

Since f^0 depends upon t'' only through $\boldsymbol{\varrho}''$, and upon $\boldsymbol{\varrho}''$ only through the energy $\mathcal{E}$ (which is a function of $\mathbf{k}$) and the temperature T (which is a function of $\mathbf{r}$) and the chemical potential (which is a function of T and the particle density n), we may write

$$\frac{df^0}{dt}=\frac{\partial f^0}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathbf{k}}\cdot\dot{\mathbf{k}}+\frac{\partial f^0}{\partial T}\frac{\partial T}{\partial \mathbf{r}}\cdot\dot{\mathbf{r}}+\frac{\partial f^0}{\partial n}\frac{\partial n}{\partial \mathbf{r}}\cdot\dot{\mathbf{r}}=\mathbf{v}\cdot\left(\frac{\partial f^0}{\partial \mathcal{E}}q\mathbf{E}+\frac{\partial f^0}{\partial T}\nabla T+\frac{\partial f^0}{\partial n}\nabla n\right). \quad (8)$$

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[1] R. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).
[2] H. Budd, Phys. Rev. **127**, 4 (1962).
[3] J. Tavernier, Compt. Rend. **255**, 120 (1962).

Heine[4] and Suzuki[5] consider the equation which one would obtain from (6a) by inserting (8), setting $\nabla T=\nabla n=\delta f(\varrho',t')=0$, and removing the factor $\partial f^0/\partial \mathcal{E}$ from under the integral sign. Clearly, the last operation leaves one with a solution of the *linearized* time-dependent Boltzmann equation, but not of the exact equation. Heine claims to prove that it is a solution of the exact equation, but his proof contains an error; however, if $\partial f^0/\partial \mathcal{E}$ is put back under the integral sign, then his proof goes through verbatim. Suzuki uses this approximate equation to discuss boundary value problems, and provides references to other recent work based on this equation. Budd[2] shows (for a special case) that the equation obtained from (7a) by inserting (8), setting $\nabla T=\nabla n=0$, and removing the factor $\partial f^0/\partial \mathcal{E}$ from under the integral sign is a solution of the linearized time-*independent* Boltzmann equation, but not of the exact equation.

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Discussion of the Dynamical Equations for the Regge Parameters*

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The dynamical equations for the Regge parameters in the relativistic, many-channel case are reduced into a single integral equation for $\mathrm{Im}\,\alpha(t)$, which is convenient to solve numerically. The solution of this integral equation is shown to exist and to be unique, after one subtraction constant for each of the functions $\alpha(t)$ and $r_{ij}(t)$, and the location of zeros of the residue functions $r_{ij}(t)$ are supplied, provided that some conditions on the subtraction constants are satisfied.

I. INTRODUCTION

It has been shown[1] that the analytic property of the Regge parameters, together with the unitarity condition, constitutes a set of equations for determining these parameters. However, many features of this set of equations, in particular, the question of what we should put in and what we can get out of them, were not well understood at that time. Neither was it realized then that inelastic two-particle intermediate states in the unitarity condition can be included, without the complication of solving some coupled integral equations.

In this paper, we shall show: (1) that the equations, with all two-particle intermediate states in the unitarity condition taken into account, can be reduced to a single integral equation which has $\mathrm{Im}\alpha(t)$ as the only unknown variable,[2] and can be solved numerically—results will be reported in a forthcoming paper[3]; (2) that the parameters of this equation will be completely specified if one subtraction constant for $\alpha(t)$ and for each of the residue functions $r_{ij}(t)$, as well as the location of zeros for $r_{ij}(t)$, are supplied; (3) that this integral equation has a unique solution if some conditions on the subtraction constants are satisfied.

These conclusions show, firstly, that the number of subtractions is not arbitrary. If we put too many restrictions on the Regge parameters by making too many subtractions, no solution for $\mathrm{Im}\alpha(t)$ would exist, while if we make too few subtractions, the solution for $\mathrm{Im}\alpha(t)$ would not be unique. Secondly, the location of the zeros of $r_{ij}(t)$ cannot be determined dynamically, but have to be supplied as input parameters. Therefore, the fact that $r_{ij}(t)$ of the Pomeranchuk trajectory vanishes at the point $\alpha_p=0$ does not follow as a dynamical consequence of our equation, but is a boundary condition itself. Whether the zeros of $r_{ij}(t)$ can be determined, once the approximate unitarity condition used here is replaced by the exact form, still awaits investigation. However, it is a consequence of analyticity and factorization for $r_{ij}(t)$ that all $r_{ij}(t)$ of the same trajectory should have the same zeros, if the possibility of double zero is ignored. The factorization law gives[4]

$$r_{ij}(t)r_{ji}(t)=r_{ii}(t)r_{jj}(t)\,,$$

and if time-reversal invariance holds,

$$r_{ij}(t)=r_{ji}(t)\,,$$

then

$$r_{ij}(t)=[r_{ii}(t)r_{jj}(t)]^{1/2}. \tag{1}$$

If $r_{ii}(t)$ has a first-order zero at z_0 and $r_{jj}(t)$ does not, then z_0 is a square-root branch point for $r_{ij}(t)$, in contradiction of the analytic property of $r_{ij}(t)$. Therefore, we should put in the same zeros for all $r_{ij}(t)$ in the dynamical equations.

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